Random and deterministic walks on lattices

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Abstract.
Random walks of particles on a lattice are a classical paradigm for the microscopic mechanism underlying diffusive processes. In deterministic walks, the role of space and time can be reversed, and the microscopic dynamics can produce quite different types of behavior such as directed propagation and organization, which appears to be generic behaviors encountered in an important class of systems. The various aspects of classical and not so classical walks on latices are reviewed with emphasis on the physical phenomena that can be treated through a lattice dynamics approach.

TEMPORAL VERSUS SPATIAL DISPERSION

One of the fundamental physical paradigms, applicable to a wide variety of physical processes, is that of *spatial diffusion*. The text-book example is a random walker on a one-dimensional lattice (see, e.g. [1]) where at each tick of the clock, the walker takes a step either to the left or to the right, the direction being chosen randomly with equal probabilities. One then asks what is the probability that the walker be at a given position after a given time. If the walker starts at a known point, the answer is a binomial distribution which, in the continuum limit, becomes a Gaussian. The variance of the Gaussian grows with time so that the localization of the walker decreases, and we say that the walker disperses. If the probability for the walker to step in one direction is greater than that for the opposite direction, then the walker propagates in the direction of higher probability and will eventually visit each site of the lattice in that direction. The typical spatial diffusive behavior is then manifested in the continuum limit as a Gaussian about a most-likely position which moves at a constant velocity. However, there are a number of situations in which, instead of asking where the walker would be after a given time (long with respect to the duration of an elementary time step), it is more natural to ask how long it will take to reach a given point, at some large distance from the starting position (large compared to the unit length covered during the elementary time step). More precisely for a stochastic process, one then asks what is the distribution of times taken to reach that point, a question related to the problem of *first-passage processes* [2].

Everyday examples involve processes in which the goal is to arrive at a given point: for example, the marathon (wherein we ask for the distribution of finishing times), certain financial instruments, such as stock options (wherein we ask for the distributions of times needed for an asset to reach a certain value), traffic-flow (wherein we ask for the distribution of arrival times at destination), and packet transport over the internet. A more technical example is the behavior of certain cellular automata which model the motion of a particle on a substrate of scatterers (in 1 or 2 dimensions) where, for

certain types of scatterers, the particle ends up propagating along a particular channel, and, again, the first-passage time is the physical quantity of interest. A paradigm for this type of behavior is the automaton known as "Langton's ant" [3, 4] which is described below. The interesting fact is that even a simplified one-dimensional version of that automaton shows the same type of behavior. This 1-D model is the analogue of the one-dimensional random walker, with the important difference that the roles of space and time are reversed: for large distances, the distribution of first-passage times is Gaussian in the time variable with a variance that grows with increasing distance from the origin. In analogy with spatial diffusion that occurs in ordinary diffusive phenomena, this generic behavior is called *temporal diffusion*. We will show how starting form a simple model, a general *first-visit equation* is obtained which in the hydrodynamic limit yields the *propagation-dispersion equation* (PDE), the analogue of the classical advection-diffusion equation, and how this PDE further generalizes propagation and dispersion processes.

FIRST VISIT EQUATION

The automaton known as "Langton's ant" [3] lives in a two-dimensional universe spanned by the square lattice with checker board parity, so defining H sites and V sites. A particle (the ant) moves from site to site (by one lattice unit length) in the direction given by an indicator. One may think of the indicator as a 'spin' (up or down) defining the state of the site. When the particle arrives at a site with spin up (down), it is scattered to the right (left) making an angle of $+\pi/2$ ($-\pi/2$) with respect to its incoming velocity vector. But the particle modifies the state of the visited site (spin up $\Longleftrightarrow$ spin down) so that on its next visit, the particle is deflected in the direction opposite to the scattering direction of its former visit. Thus the particle entering from below a H site with spin up is scattered East, and on its next visit to that same site (now with spin down), if it arrives from above, it will be scattered East again, while if it arrives from below, it will be scattered West. Similar reasoning shows how the particle is scattered North or South on V sites.

At the initial time, all sites are in the same state (all spins up or down), and the position and velocity direction of the particle are fixed, but arbitrary. So if we paint the sites black or white according to their spin state and we start with say an all white universe, then, as the particle moves, the visited sites turn alternately black and white depending on whether they are visited an odd or even number of times. This color coding offers a way to observe the evolution of the automaton universe. The particle starts exploring the universe by first creating centrally symmetric transient patterns (see figures in references [3]), then after about 10 000 time steps (9977 to be precise), it leaves a seemingly 'random territory' to enter a 'highway' (see Fig.1) showing a periodic pattern. The "disordered" phase is not what a random walk would produce: the automaton is deterministic and its rules create correlations between successive states of the substrate, so also between successive positions of the particle. The power spectrum computed from the particle position time correlation function measured over the first 9977 time steps goes like $\sim \nu^{-\zeta}$ with $\zeta \simeq 4/3$. In the ordered phase (the 'highway'), the power spectrum

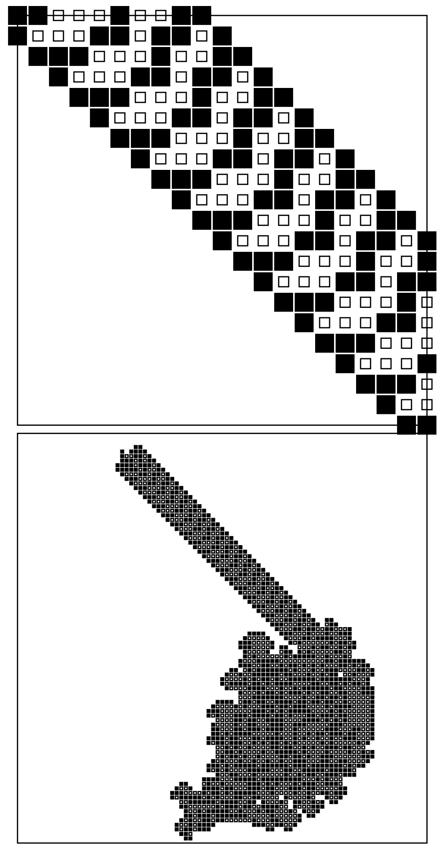

FIGURE 1. Langton's ant trajectory after 12,000 automaton time steps. The upper box is a blow-up of the highway showing the periodic pattern. Sites with open squares and dark squares have opposite spin states (up and down).

shows a peak at $\nu = 1/104$ with harmonics. Indeed in the highway, the particle travels with constant propagation speed: $c = 2\sqrt{2}/104$ (in lattice units).

Because of the complexity of the dynamics on the square lattice, Grosfils, Boon, Cohen, and Bunimovich [5] developed a one-dimensional version of the automaton for which they provided a complete mathematical analysis also applicable to the two-dimensional triangular lattice. In the one-dimensional case, the particle moves in the direction of its velocity vector with probability q and in the opposite direction with probability $(1-q)$, the direction being dictated by the "spin" of the lattice site, which

is then reversed after the passage of the particle. The mean-field equation describing the microscopic dynamics of the particle with the general condition that the spins at the initial time are randomly distributed on the lattice, reads [5]

$$f(r+1,t+1) = q\,f(r,t) + (1-q)\,f(r,t-2)\,. \tag{1}$$

Here $f(r,t)$ is the single particle distribution function, i.e. the probability that the particle visits site r *for the first time* at time t, and q is the probability that the immediately previously visited site along the propagation strip (the highway) has initially spin up, i.e. the probability that the particle be scattered along the direction of its velocity vector when arriving at the scattering site at $r-1$. [1] An important result follows that can be formulated as a theorem [5] : a particle moving from site to site in a one-dimensional lattice fully occupied with flipping scatterers (spins), propagates in one direction, independently of the initial distribution of the spins on the lattice.

There are two points of particular interest here.

(i) First one notices that Eq. (1) has the same structure as the equation for the one-dimensional random walk [1]

$$g(r+1,t+1) = q g(r,t) + (1-q)\,g(r+2,t)\,, \tag{2}$$

except that the in the second term on the r.h.s. of Eq. (1) one has $t-2$ whereas in the random walk equation (2) one has $r+2$; this increment transfer between space and time makes a crucial difference as we shall see below.

(ii) Eq.(1) is a particular case of a general equation [6]. To see this, consider a walker on a one-dimensional lattice and let $f(t/\delta t; r/\delta r)$ be the probability that it takes $t/\delta t$ time steps to reach the lattice position $r/\delta r$, given that the walker is at the origin at time $t=0$. Whatever the microscopic dynamics, we assume that we are given, or can work out, the set of probabilities $\{p_j(r)\}_{j=1}^{\infty}$ that the time between the first visit of the lattice site $r/\delta r$ and the first visit of the next position, $r/\delta r+1$, is $\mu_j \delta t$. Conceptually, these represent the probabilities of various waiting times from the first visit of lattice site $r/\delta r$ until the first visit to $r/\delta r+1$, i.e. the distribution of single-step waiting times. It is then clear that the probability that it takes time t for the walker to reach the lattice site $r+\delta r$ is equal to the probability that it takes time t to reach lattice site r and that the waiting time is zero, plus the probability that it takes time $t-\delta t$ to reach site r and that the waiting time is δt, plus the probability that the waiting time is $2\delta t$, . . . so that the master equation is

$$f(t/\delta t; r/\delta r+1) = \sum_{j=0}^{n} p_j(r)\, f(t/\delta t-\mu_j; r/\delta r)\,, \tag{3}$$

or

$$f(t; r+\delta r) = \sum_{j=0}^{n} p_j(r)\, f(t-\tau_j; r)\,. \tag{4}$$

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[1] A similar equation holds for the two-dimensional triangular lattice and these equations were shown to yield exact solutions for propagative behavior (corresponding to an ordered phase of the lattice) in the classes of models considered by Grosfils *et al.* [5].

This is the *first visit equation* [6] where p_j is the probability that it takes time $\tau_j = \mu_j \, \delta t$ for the particle to propagate from r to $r + \delta r$, i.e. τ_j is the time delay between two successive first visits on the propagation strip for the path with probability p_j. The sum is over all possible time delays, weighted by the probability p_j, and n can be finite [6] or infinite [7], the two formulations being equivalent, depending on whether the distribution of the delays is contained either in the τ_j's or in the p_j's.

For one particular realization, the successive time delays are set by a given spatial configuration of the time delayers, and the time taken by the particle to perform a displacement from r to $r + \delta r$ depends on that configuration. For an ensemble of realizations, the distribution function of the time delays defines the average displacement time

$$\langle \tau \rangle = \sum_{j=0}^{n} p_j \, \tau_j = \sum_{j=0}^{n} \mu_j \, p_j \, \delta t = \langle \mu \rangle \, \delta t \, , \tag{5}$$

and the variance

$$\begin{aligned} \langle \tau^2 \rangle - \langle \tau \rangle^2 &= \left\{ \sum_{j=0}^{n} \mu_j^2 \, p_j - [\sum_{j=0}^{n} \mu_j \, p_j]^2 \right\} (\delta t)^2 \\ &= (\langle \mu^2 \rangle - \langle \mu \rangle^2) \, (\delta t)^2 \, , \end{aligned} \tag{6}$$

where $\mu_j = \tau_j / \delta t$ is the number of time steps during the time delay τ_j. The general condition on the p_j distribution is that its moments be finite. For specific lattice dynamics (such as for the 1-D model described above) μ_j is known analytically and the moments can be computed explicitly. For instance, one can then show that the propagation velocity of Langton's ant is $c = \delta r / \langle \tau \rangle = 2\sqrt{2}/104$ [6].

PROPAGATION-DISPERSION EQUATION

The systems that we are discussing exhibit two time scales which correspond to (i) a propagation process characterized by the average time necessary to complete a finite number of displacements $r/\delta r$

$$\mathrm{E}[t_r] = \langle \mu \rangle \, r \, \frac{\delta t}{\delta r} \, , \tag{7}$$

and (ii) the dispersion around this average value characterized by the variance

$$\mathrm{Var}[t_r] = (\langle \mu^2 \rangle - \langle \mu \rangle^2) \, (\delta t)^2 \, \frac{r}{\delta r} \, . \tag{8}$$

For finite r, these are finite quantities. Correspondingly we define the following quantities that will be used in the hydrodynamic limit of Eq.(4)

$$\frac{1}{c} = \langle \mu \rangle \, \frac{\delta t}{\delta r} \, , \tag{9}$$

and

$$\gamma = (\langle \mu^2 \rangle - \langle \mu \rangle^2) \frac{(\delta t)^2}{\delta r} . \tag{10}$$

c ($\neq 0$) will be identified as the propagation speed and γ (≥ 0) will be identified as the dispersion coefficient.

The hydrodynamic limit, i.e. for $r/\delta r \gg 1$, can be obtained by multi-scale expansion starting from Eq.(4) or by the generating function method (with application of the central limit theorem) [2]. With the multi-scale expansion, one obtains the *propagation-dispersion equation* [8]

$$\partial_r f(r,t) + \frac{1}{c} \partial_t f(r,t) = \frac{1}{2} \gamma \partial_t^2 f(r,t) , \tag{11}$$

and with the other method, one obtains its solution [9]

$$f(r,t) = \sqrt{\frac{1}{2\pi}} (\gamma r)^{-\frac{1}{2}} \exp\left(-\frac{(t - \frac{r}{c})^2}{2\gamma r} \right) , \tag{12}$$

with the the initial condition that at the origin, say at $r = 0$, $f(0,t) = \delta(t)$. Note that this condition is not restrictive in that, if the initial distribution is given by some function $f(t; r = 0) = f_0(t)$, the solution is the result (12) convoluted with $f_0(t)$.

It is clear from (11), that c is a propagation speed, and γ is a transport coefficient expressing dispersion in time (instead of space like in the classical Fokker-Planck equation for diffusion). Equation (11) is the propagation-dispersion equation governing the first-passage distribution function of a propagating particle subject to time delays. Figure 2 illustrates these results.

Note that for the biased random walker in the continuum limit, the exact first passage time distribution is known [1, 10] to be

$$f(t;r) = \frac{r}{\sqrt{2\pi D} t^{3/2}} \exp\left(-\frac{(r-ct)^2}{2Dt} \right) , \tag{13}$$

where D is the spatial diffusion coefficient. The difference between this expression and (12) is due to the fact that the latter is an approximation which is only valid for large r. In this regime, the exact result only gives a non-zero probability for $(r-ct)^2/2Dt = \mathcal{O}(1)$ which implies $ct = r + \mathcal{O}\left(\sqrt{2Dr/c}\right) = r\left(1 + \mathcal{O}\left(\sqrt{2D/cr}\right)\right)$. So, for large r we can use this approximation to write the exact distribution as

$$f(t;r) = \frac{c^{3/2}}{\sqrt{2\pi D} r^{1/2}} \exp\left(-\frac{(r-ct)^2}{2Dr/c} \right) \left(1 + \mathcal{O}\left(\sqrt{2D/cr}\right)\right) , \tag{14}$$

which, with $D/c^3 = \gamma$, agrees with the large-distance result (12). We emphasize that Eq.(13) is exact in the continuum limit, i.e. for vanishing δr and δt, whereas the only

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[2] The reader is referred to [8, 9] for the analytical computations.

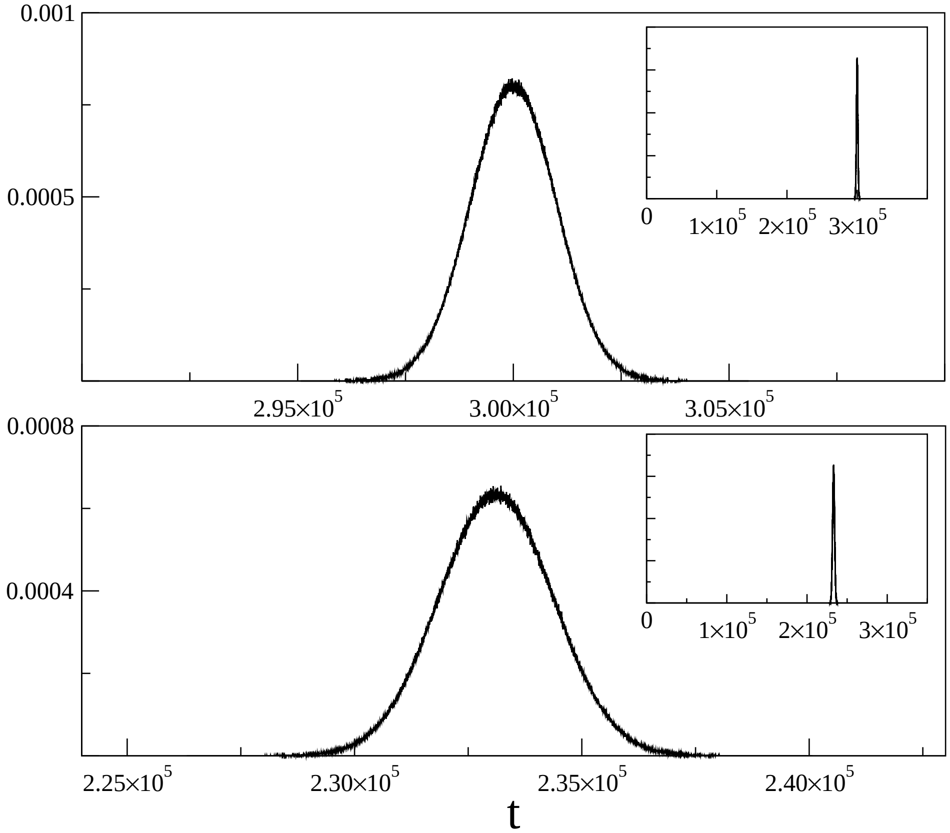

FIGURE 2. Probability distribution $f(r = 3\times 10^4, t)$ based on general equation (4). (a) time delays equally distributed for $j = 0, 1, \cdots, 9$, with $p_j = 0.1$; $c = 0.1$ and $\gamma = 33$; half-width $= \sqrt{2\gamma r} \simeq 1.41\times 10^3$. (b) time delays exponentially distributed: $p_j = C\exp -\beta j$, with $j = 0, 1, ..., 9$, $\beta = 0.25$, and $C = [\sum_{j=0}^{9} j]^{-1} = 1/45$; $c = 0.128$ and $\gamma = 52.7$; half-width $= \sqrt{2\gamma r} \simeq 1.78\times 10^3$. The numerical simulation data and the analytical expression (Eq.(12); solid line, not visible) coincide perfectly. Insets show large scale representation. Space and time are in automaton units.

restrictions on the general result (12) are that r is large and that the first two moments of the elementary waiting time distribution, $\langle\tau\rangle$ and γ, exist. The latter condition precludes the limit of the symmetric random walker, $c \to 0$, for which $\langle\tau\rangle = \delta r/c$ diverges (see (5) and (9)).

TEMPORAL DIFFUSION

It follows from Eqs.(8) and (10), that the dispersion coefficient γ is given by

$$\gamma = \frac{\langle t_r^2\rangle - \langle t_r\rangle^2}{r} \tag{15}$$

which, for large r, is reminiscent of the classical expression for the diffusion coefficient: $D = \lim_{t\to\infty}\langle r^2(t)\rangle/2t$. Comparison of the two expressions shows interchange of space

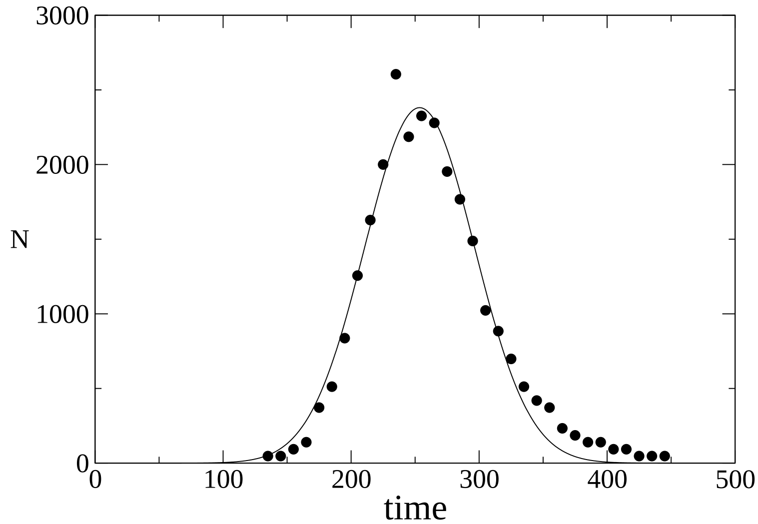

FIGURE 3. New York marathon (1996): distribution of arrival times (in minutes; N = number of runners). Data (black dots) and Gaussian fit (solid curve). The skewness indicates that all runners are not subject to the same waiting time probability distribution.

and time, and measurements of the variance $\langle t_r^2 \rangle - \langle t_r \rangle^2$ should show a linear dependence in terms of the distance with a slope equal to γ in the same way as the diffusion coefficient is obtained as the slope of the mean-square displacement versus time in the long-time limit.

An interesting case is the experimental study of the diffusion of a single particle in a 3-D random packing of spheres [11] which describes the motion of a particle through an idealized granular medium. Here one measures particulate transport and 'dispersivity' which corresponds precisely to the quantity γ. The experimental data presented in [11] show that the mean square transit time of the particle through the medium is a linear function of the mean transit time (Figs.10 and 11 in [11]) itself a linear function of the percolating distance (Fig.2 in [11]). This observation is a clear experimental illustration of the feature of Eq.(15). This experimental study also shows that the particle transit time is Gaussianly distributed in time (see Fig.9 in [11]) in accordance with the solution (12) of Eq.(11) (see Fig.2).

A popular example where the concept of temporal diffusion is obviously relevant is the Marathon. Each runner can be viewed as a particle moving on a one-dimensional path - the race track - starting from a given origin and heading towards the finish line, with time delays along its trajectory. Each such trajectory represents one realization of the dynamics, which generates a distribution approximated by a Gaussian (as shown in Fig.3) whose first moment is the average time of arrival ($\langle \tau \rangle \simeq 255$ min) with $c = 42.195 \times 60 / \langle \tau \rangle \sim 10$ (km/hour), and whose second moment gives a measure of the dispersion coefficient $\gamma \simeq .055$ min^2/m.

The dispersion coefficient can be further expressed in terms of the fluctuations in

the *local propagation velocity* $v(r)$, a quantity with average value c. In fact it is the reciprocal local velocity which is physically relevant: it is the time taken by the particle to propagate from position r to $r+\delta r$ (divided by δr). Then indeed

$$\langle t_r \rangle = \langle \int_0^r dr' \frac{1}{v(r')} \rangle = \int_0^r dr' \langle \frac{1}{v(r')} \rangle = \frac{r}{c}, \tag{16}$$

which is consistent with the definition of the propagation speed. It is then easy to compute the variance in terms of the reciprocal velocity fluctuations $\delta v^{-1}(r) = v^{-1}(r) - \langle v^{-1} \rangle = v^{-1}(r) - c^{-1}$,

$$\langle t_r^2 \rangle - \langle t_r \rangle^2 = \int_0^r dr' \int_0^r dr'' \langle \delta v^{-1}(r')\, \delta v^{-1}(r'') \rangle . \tag{17}$$

If the dynamics of the propagating particle is such that the correlation function on the r.h.s. of (17) is δ-correlated, i.e. $\langle \delta v^{-1}(r')\, \delta v^{-1}(r'') \rangle = \phi_0\, \delta(\frac{r'}{\xi} - \frac{r''}{\xi})$ with $\phi_0 = \langle (\delta v^{-1})^2 \rangle = \langle \frac{1}{v^2} \rangle - \frac{1}{c^2}$, and where ξ is the elementary correlation length, it follows from (15) and (17) that

$$\gamma = \xi (\langle v^{-2} \rangle - c^{-2}), \tag{18}$$

that is γ is the covariance of the reciprocal velocity fluctuations multiplied by the correlation length. This result is analogous to Taylor's formula of hydrodynamic dispersivity which is expressed as the product of the covariance of the velocity fluctuations with a characteristic correlation time [12]. Here γ is the *temporal* dispersivity.

In classical advection-diffusion phenomena, the control parameter is the Péclet number $P = UL/2D$, where U denotes the mean advection speed, L, the characteristic macroscopic length, and D, the diffusion coefficient (see e.g. [13]). The analogue for propagation-dispersion follows by casting Eq.(11) in non-dimensional form

$$\partial_{\mathtt{r}} f(r,t) + \partial_{\mathtt{t}} f(r,t) = B^{-1} \partial_{\mathtt{t}}^2 f(r,t)\, ; \quad B = \frac{2T}{\gamma c} . \tag{19}$$

Here $\mathtt{r}$ and $\mathtt{t}$ are the dimensionless space and time variables: $\mathtt{r} = r(cT)^{-1}$ and $\mathtt{t} = t\,T^{-1}$, where T is a characteristic macroscopic time. B is the control parameter for propagation-dispersion: it is a measure of the relative importance of propagation with respect to dispersion. Indeed, $B = \frac{2T}{\gamma c} = \frac{2T^2}{\gamma} \frac{1}{cT} = L_D/L_P$, i.e. the ratio of the characteristic dispersion length L_D to the characteristic propagation length L_P. At high values of B, i.e. $L_D \gg L_P$, the distribution function is very narrow, and transport over large distances ($r \geq cT$) is dominated by propagation.

GENERALIZED PROPAGATION-DISPERSION

There are two important generalizations of the propagation-dispersion equation. The first generalization is for temporal diffusive behavior in inhomogeneous systems, i.e. for processes where the waiting time probabilities depend on the location of the particle.

The p_j's are then space dependent, and the propagation-dispersion equation becomes [9]

$$\frac{\partial}{\partial r} f(t,r) + \frac{1}{c(r)} \frac{\partial}{\partial t} f(t,r) = \frac{1}{2} \gamma(r) \frac{\partial^2}{\partial t^2} f(t,r) \tag{20}$$

with

$$\frac{1}{c(r)} \quad = \quad \frac{\partial}{\partial r} \tau(r) \,, \tag{21}$$

and

$$\gamma(r) \quad = \quad \frac{\partial}{\partial r} \sigma^2(r) \,, \tag{22}$$

where

$$\tau(r) = \delta t \sum_{k=1}^{r/\delta r} \sum_{j=0}^{\infty} j p_j((k-1)\,\delta r) \,, \tag{23}$$

and

$$\sigma^2(r) = (\delta t)^2 \sum_{k=1}^{r/\delta r} \left[\sum_{j=0}^{\infty} p_j((k-1)\,\delta r)\, j^2 - \left(\sum_{j=0}^{\infty} p_j((k-1)\,\delta r)\, j \right)^2 \right] . \tag{24}$$

The solution of Eq.(20) reads

$$f(t,r) = \int_{-\infty}^{\infty} \sqrt{\frac{1}{2\pi\sigma^2(r)}} \exp\left(-\frac{(t - t' - \tau(r))^2}{2\sigma^2(r)} \right) f_0(t')\, dt' \,, \tag{25}$$

where $f_0(t) = f(t; r=0)$. Buminovich and Khlabystova [7] have studied models similar to the one-dimensional model described earlier in this chapter, but in which the scatterers only change state after multiple scattering events [14]. In this case, the distribution of elementary waiting times becomes dependent on the lattice position, and the propagation speed and dispersion coefficient acquire a spatial dependence. Thus, while the distributions of first passage times are still Gaussian, they are not "diffusive" in the usual sense since the inverse propagation speed and dispersion coefficient are not constants (Eqs.(20-22)).

The phenomena described so far are for cases where the variance of the elementary time-delay processes exist. The second generalization is for the interesting class of similar, but more complex processes which are described by power-law distributions which do not possess second moments, e.g. $p_j \sim \tau_j{}^{-(1+\alpha)}$, in which case, for $0 < \alpha \leq 1$, the distribution appearing in the central limit theorem is no longer Gaussian (see e.g. [15] and the appendix in [16]).

For the Pareto distribution

$$p(t) = \Theta(t - t_0) \frac{\alpha t_0^{\alpha}}{t^{1+\alpha}} \quad ; \quad 0 < \alpha < 2,\ \alpha \neq 1 \,, \tag{26}$$

one can show that the propagation-dispersion equation becomes

$$\frac{\partial}{\partial r} f_\alpha(t;r/\delta r) = \left[\frac{\alpha t_0}{(1-\alpha)\,\delta r}\frac{\partial}{\partial t} - t_0^\alpha \frac{\Gamma(1-\alpha)}{\delta r}\frac{\partial^\alpha}{\partial t^\alpha}\right] f_\alpha(t;r/\delta r) \ , \tag{27}$$

where the fractional derivative can be defined through the Fourier transformation

$$\frac{\partial}{\partial t^\alpha} f_\alpha(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega (-\imath\omega)^\alpha \exp(-\imath\omega t)\tilde{f}(\omega) \ . \tag{28}$$

Equation (27) is the fractional propagation-dispersion equation. For the special case $\alpha = 1/2$, the equation becomes

$$\frac{\partial}{\partial r} f_{1/2}(t;r/\delta r) = \left[\frac{t_0}{\delta r}\frac{\partial}{\partial t} - \frac{\sqrt{\pi t_0}}{\delta r}\frac{\partial^{1/2}}{\partial t^{1/2}}\right] f_{1/2}(t;r/\delta r) \ , \tag{29}$$

whose solution reads

$$f_{1/2}(t;r/\delta r) = \frac{1}{2}\sqrt{t_0}\frac{\left(\frac{r}{\delta r}\right)}{\left(t+\frac{r}{\delta r}t_0\right)^{3/2}} \exp\left(-\frac{t_0\pi\left(\frac{r}{\delta r}\right)^2}{4\left(t+\frac{r}{\delta r}t_0\right)}\right) \ . \tag{30}$$

The fractional equation (27) should be contrasted with the fractional Fokker-Planck equation which has been studied extensively for anomalous spatial diffusion [22]. The fractional propagation-dispersion equation (27) is new and is expected to be appropriate for the description and the analysis of non-Gaussian (anomalous) temporal diffusive processes.

COMMENTS

There is an algebraic similarity in the structure of the propagation-dispersion equation (11) and that of the classical advection-diffusion equation [1] which can be formally transformed into each other by interchanging space and time variables. It should be clear that the two equations describe different, but complementary aspects of the dynamics of a moving particle. Solving the propagation-dispersion equation answers the question of the time of arrival and of the time distribution around the average arrival time in a propagation process. It is also legitimate to ask the complementary question “where should we expect to find the particle after some given time ?” which should be long compared to the elementary time step, but short with respect to the average time of arrival. We will then observe spatial dispersion around some average position which can be evaluated from the solution of the advection-diffusion equation. This observation stresses the complementarity of the two equations.

Because the propagation-dispersion equation describes the space-time behavior of the *first passage* distribution function $f(r,t)$, i.e. the probability that a particle be for the first time at some position, it describes transport where a first passage mechanism plays an important role. So the equation should be applicable to the class of front-type

propagation phenomena where any location ahead of the front will necessarily be visited, the question being: *when* will a given point be reached?

Besides the examples discussed above, temporal diffusion is also encountered in shock propagation in homogeneous or inhomogeneous media [17] or packet transport in the Internet [18]. As the propagation-dispersion equation is for the first-passage time distribution, it should also be suited for the description of transport driven by an input current in a disordered random medium [19]. In the area of traffic flow, there are typical situations where cars moving on a highway from location A to location B, are subject to time delays along the way, and – with the assumption that all cars arrive at destination – one wants to evaluate the time of arrival [20]. Financial series as in the time evolution of stock values are another example [21]: over long periods of time (typically years) one observes a definite trend of increase of, for instance, the value of the dollar. So any preset reachable value will necessarily be attained, the questions being: when? and what is the time distribution around the average time for the preset value? While the classical question is: after such or such period of time, which value can one expect?, there might be instances where the reciprocal question should be considered. Because of the generality of the propagation-dispersion equation, it should be expected that, either in its simple form (11) or in its generalized forms, (20) and (27), the equation be applicable to a large class of first-passage type problems in physics and related domains.

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